Aerodynamic Response of Airfoils in Sinusoidal Oblique Gust

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A linearized theory has been developed to provide check results for advancing numerical calculations of unsteady lift and moment of airfoils in subsonic sinusoidal oblique gusts The airfoils are thin flat plates of infinite span and may be in cascade The gust response curves for varying flow Mach number and spanwise gust wave number are presented The obliqueness of the gust wave front with respect to the airfoil leading edge is found to be very influential upon the aerodynamic lift and moment

Nomenclature

c	= airfoil chord
C_0	= speed of sound
$\mathcal{C}_L$ $\mathcal{C}_M$	= lift and moment coefficients
$f(x)$	= lift distribution function
h	= displacement of bending vibration
I_α	= moment of inertia for torsional vibration
$\mathcal{L}$	= total fluctuating lift
m	= number of cascade airfoils or mass per unit span
M_0	= mean flow Mach number
$\mathcal{M}$	= total fluctuating moment
p	= pressure perturbation
$\mathcal{P}$	= kernel function to obtain p
s	= cascade pitch
t	= time
v	= velocity perturbation normal to airfoil surface
V	= mean flow speed
V	= velocity perturbation vector
$\mathcal{V}$	= kernel function to obtain v
$(x\ y\ z)$	= coordinates
x_g x_η	= mass center and axis of rotation
α	= displacement of torsional vibration
β	$=\sqrt{1-M_0^2}$
γ	= ratio of specific heats
γ_h γ_α	= logarithmic decrement of bending and torsional vibration due to aerodynamic damping
θ	= cascade stagger angle
κ	$=(M_0/\beta^2)(\omega/V)$
λ	= spanwise wave number
μ	$=\kappa M_0$
ρ	= density perturbation
σ	= interblade phase angle
ω	= angular frequency

Introduction

UNSTEADY aerodynamic lift and moment upon airfoils due to convected upwash disturbance such as gusts or turbulence are very important design parameters for the safety and control of an aircraft That is for modern fans and compressors as well where the gusts will be inlet distortion or viscous wakes from rotor and/or stator blades which interact with moving rear blades

The response of thin airfoils to a sinusoidal gust in a plane flow has been thoroughly investigated since Sears[1] derived his famous analytical function Recent studies concern the more general cases that include 1) the airfoil thickness and steady loading and 2) the three dimensional gust and airfoil interaction pattern Formulation to allow obliqueness of the gust wave fronts with respect to the airfoil span falls into this second category of the problem Graham[2] developed a lifting surface theory to cope with the latter oblique gust Filotas[3] solved the same three dimensional incompressible flow problem deriving compact expressions for the lift and pressure distribution Mugridge[4] also made a three dimensional modification to the Sears problem and obtained a closed form approximate solution Extension to include compressibility effect has been made by Graham[5] who constructed similarity rules between compressible and incompressible gust flows Johnson[6] investigated the airloads induced upon an infinite span wing by a straight free vortex which lies parallel to the wing plane Chu and Widnall[7] employed the matched asymptotic expansion method in which the inner region is assumed to be incompressible under the condition of large acoustic wavelength/chord ratio This was recast by Filotas[8] into a convenient form of the Sears function multiplied by compressibility and obliqueness'' factors

The above results are all for an isolated airfoil In the case of cascade blades sweep and circumferential tilt of the blade setting will produce the gust interaction in which the upwash on the blade surface takes the same general form of traveling waves along the blade span

In this paper therefore, a linearized approach is developed to predict the unsteady lift and moment of both isolated and cascade airfoils which operate in subsonic flow in the presence of a sinusoidal gust of the general form $\exp[i\omega(t-x/V)+i\lambda z]$ The interest of this investigation may be the interaction of such waves with airfoils of finite span which Goldstein[9] treated in a general theory and for which Atassi and Hamad[10] performed calculations of the sound generation The airfoils are here assumed for simplicity to be thin plates of infinite span with constant chord In the case of an isolated airfoil the present analysis thus becomes identical with previous work However further results of gust response curves with Mach number and spanwise wave number λ changed are presented For cascade blades on the other hand the corresponding response curves including the effect of spanwise wave number λ are not much known Therefore a case study is made and its results are presented choosing a typical cascade geometry The aerodynamic damping for cascade blades in torsional vibration is also discussed in relation to the occurrence of blade flutter

Analysis based upon such simple linearized flow models may no longer be of much practical use with the advance of sophisticated numerical calculation methods Linearized

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analysis is nevertheless a valuable first step towards the correct solution since it provides a concise and reliable tool for checking vast computer programs

Theory

The following assumptions are made in the present analysis:

1) The fluid is inviscid and compressible and the flow is isentropic

2) The airfoils are flat plates of negligible thickness of infinite span with constant chord

3) The perturbations from the uniform flow are small and possess harmonic time dependence

4) There is no steady loading on the airfoils

5) The Kutta Joukowski condition holds at the trailing edge

Upwash Velocities

The upwash normal to the airfoil surface due to a gust which is convected by the main flow may be expressed as $v=-v_\lambda \exp[i\omega(t-x/V)+i\lambda z]$ where v_λ is the gust strength at the midchord point The gust obliqueness is measured by the intersection angle Λ between the wave fronts and airfoil span (Fig 1) The gust wave number k is given by $(\omega/V)\sec\Lambda$ or $\lambda\operatorname{cosec}\Lambda$ A similar expression is obtained for the case when inlet distortion or wakes from periodic obstruction upstream interact with cascade blades The two dimensional wakes of stationary sinusoidal profiles convected by the mean flow V_s cause the upwash in the direction normal to the surface of the downstream rotor blades of the form $\exp[i\omega\{t-x/V-y\cos(\theta_s+\theta_R)/q\cos\theta_s\}]$, where $\omega=2\pi q/b_s$ q being the rotor speed (Fig 2) The interblade phase angle σ of the rotor is given by $2\pi(\nu-b_R/b_s)$ in which ν is any integer Tilting ϕ of the blades towards y direction then yields the previous form of the upwash on the blade surface where $\lambda=-\omega\cos(\theta_s+\theta_R)\sin\phi/q\cos\theta_s$ The next type of upwash is due to gust excited airfoil vibration The vibration amplitudes h and α for bending and torsion are written as $h_0 f_h(z)$ or $\alpha_0 f_\alpha(z)$ by respective mode function f_h or f_α The Fourier transform formula will be conveniently employed to yield the upwash expression of the λ component in question i e $v=(i\omega h_\lambda - V\alpha_\lambda\{1+(i\omega/V)(x-x_\eta)\})\exp[i\omega t+i\lambda z]$ (Fig 3)

In summary three kinds of upwash velocities are present They are expressed in a matrix form:

$$v/V=[1 \quad 1+i(\omega/V)(x-x_\eta) \quad \exp\{-i(\omega/V)x\}]\begin{bmatrix} i(\omega/V)h_\lambda \\ -\alpha_\lambda \\ -v_\lambda \end{bmatrix}\exp(i\omega t+i\lambda z) \tag{1}$$

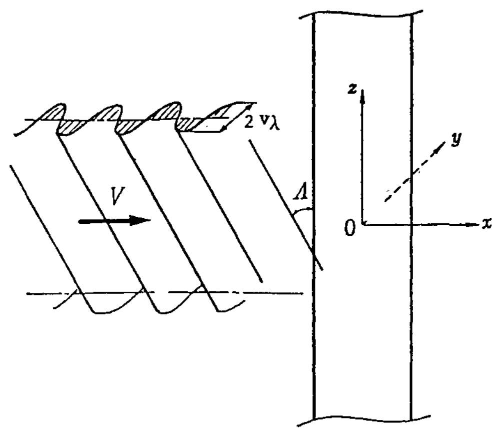


Fig 1 Sinusoidal oblique gust and airfoil interaction

Integral Equation

The linearized basic equations of continuity momentum and isentropic condition may be written as

$$\frac{d\rho}{dt}+\rho_0\nabla\ V=0 \tag{2}$$

$$\frac{dV}{dt}+\frac{1}{\rho_0}\nabla p=\frac{F}{\rho_0} \tag{3}$$

$$\frac{p}{\rho}=\gamma\frac{p_0}{\rho_0}\equiv C_0^2 \tag{4}$$

whence the convected wave equation can be derived:

$$\left[\Delta-\frac{1}{C_0^2}\frac{d^2}{dt^2}\right]p=\nabla F \tag{5}$$

The operator Δ is the Laplacian and d/dt is defined by $\partial/\partial t+V\partial/\partial x$ while F represents the lift applied to the fluid by the airfoil element in the y direction normal to the airfoil surface Putting the solution in the form of $\{p\ v\}=\{p_\lambda\ v_\lambda\}\exp[i\omega t+i\lambda z]$ the following expressions are obtained (Appendix):

$$p_\lambda(x\ y)=\int_{-c/2}^{c/2} f_\lambda(x_0)\,\mathcal{P}(x-x_0\ y|\lambda)\,dx_0 \tag{6}$$

$$v_\lambda(x\ y)/V=\int_{-c/2}^{c/2}\{f_\lambda(x_0)/\rho_0 V^2\}\mathcal{V}(x-x_0\ y|\lambda)\,dx_0 \tag{7}$$

The distribution $f_\lambda(x)$ is determined by the condition that the resulting induced velocity v_λ on and normal to the airfoil surface cancels out the given upwash v exactly thus leading to the Possio type integral equation

Lift and Moment

A scheme to employ the finite element method has been made for the purpose of solving that integral equation First

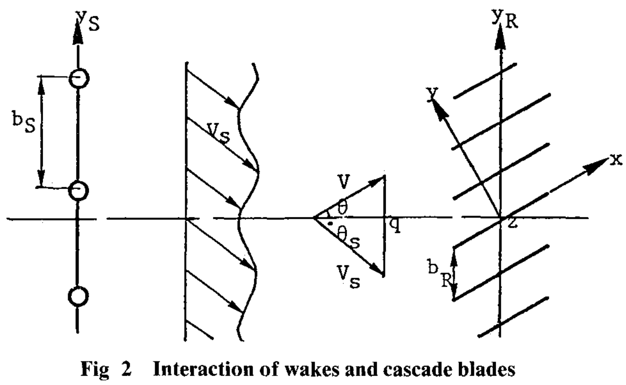


Fig 2 Interaction of wakes and cascade blades

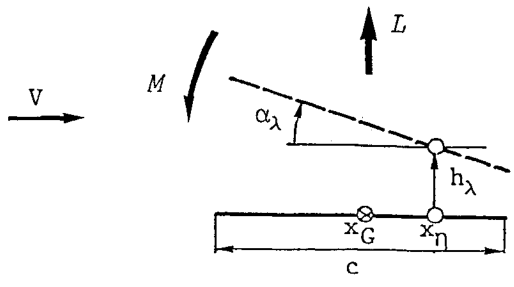


Fig 3 Airfoil vibration

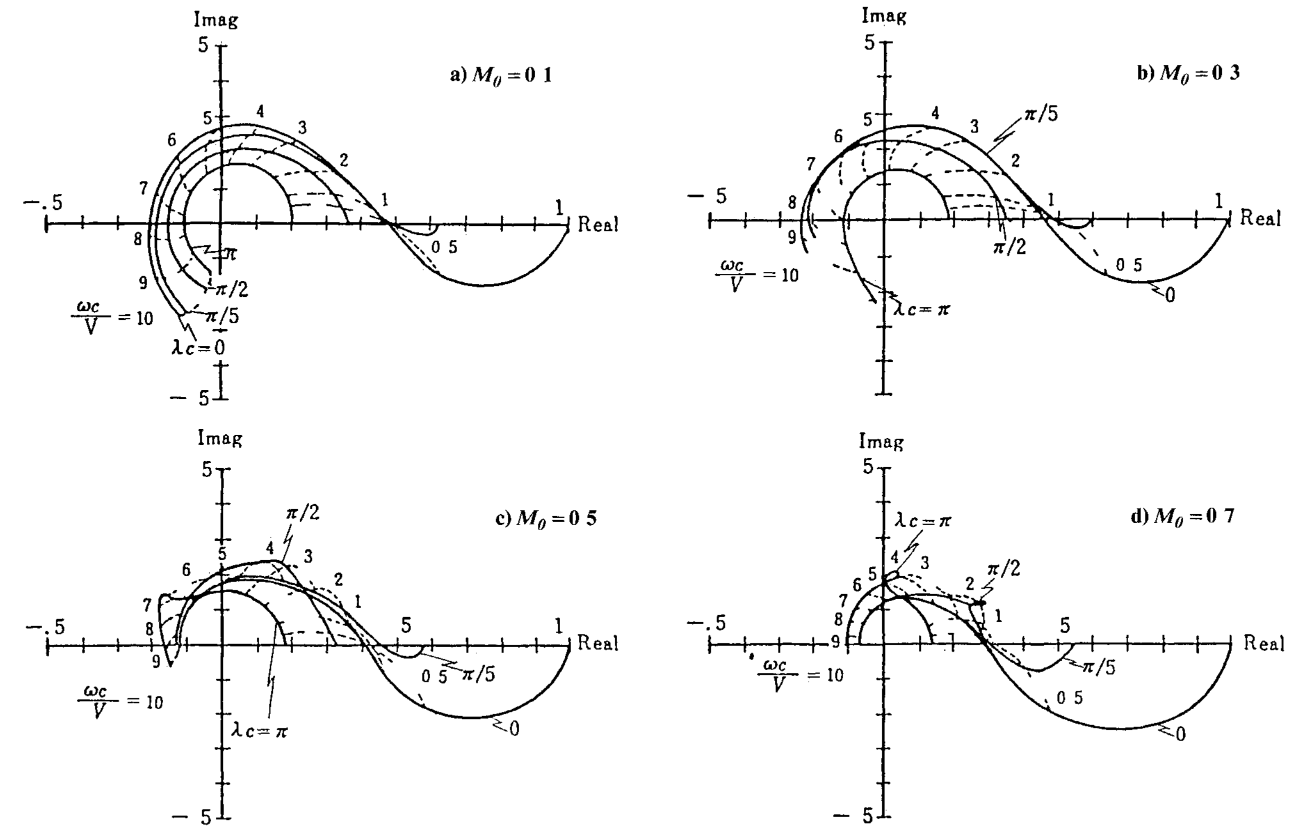


Fig 4 Effect of spanwise wave number λ upon lift coefficient at various Mach numbers

the following coordinate transformation is made $\{x\ x_0\} = -(c/2)\{\cos\psi\ \cos\epsilon\}$ so that the integral range becomes 0 to π This can be divided into M number of elements At the ith element extending from ϵ_i to ϵ_{i+1} the coordinate ϵ is related to the local coordinate $\zeta(-1 \le \zeta \le 1)$ by $\epsilon^{(i)} = \{(1-\zeta)/2\}\epsilon_i + \{(1+\zeta)/2\}\epsilon_{i+1}$ while the unknown lift distribution is assumed to be given by $f(\zeta)/(\rho_0 V^2) = \{(1-\zeta)/2\}f_i + \{(1+\zeta)/2\}f_{i+1}$, and f_i and f_{i+1} represent the values of $f(\zeta)/\rho_0 V^2$ at both ends of the element Imposition of the Kutta Joukowski condition makes it necessary to modify the latter into $f(\zeta)/(\rho_0 V^2) = \{cot(\epsilon^{(i)}/2) - cot(\epsilon_2/2)\}f_1 + \{(1+\zeta)/2\}f_2$ or $\{(1-\zeta)/2\}f_M$ for the elements including the leading or trailing edge respectively Evaluating Eq (7) on the airfoil surface $y=0$ then yields a matrix relationship between the upwash and lift distribution On specifying the upwash at M number of points over the airfoil chord local lift force f_i can be obtained Lift $\mathcal{L}$ and moment $\mathfrak{M}$ (positive for nose down) acting on the airfoil are the sum of the con tribution from those local lift forces The lift and moment coefficients are thus defined by

$$\mathcal{C}_L = \mathcal{L}/\{\pi\rho_0 V(v/\beta)c\} \tag{8}$$

$$\mathcal{C}_M = \mathfrak{M}/\{\pi\rho_0 V(v/\beta)c^2\} \tag{9}$$

The coefficients are dependent upon ω M_0 λ, x_η and in addition upon cascade geometry (s/c θ) and interblade phase angle σ for cascade blades

Results

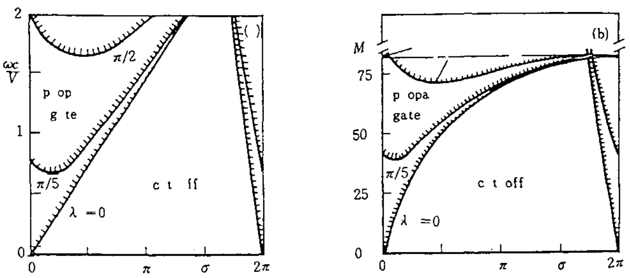
Accuracy of the present method has been checked by comparing its results with the following previous work with which excellent agreement has been obtained; 1) Sears func tion up to the frequency parameter $\omega c/V = 10$ 2) Graham s[2] incompressible results for oblique gust 3) aerodynamic coefficients of a flat airfoil by Timman et al [11] up to $\omega c/V = 3\ 5$ with Mach number changed and 4) the aerodynamic transfer function by Chu and Widnall[7] over the whole Mach number range In the latest work the phase tendency of the hyperbolic region showed the closer agreement with Johnson s[6] solution

Figure 4 shows the lift function with varying spanwise wave number at various Mach numbers When the Mach number is fairly small (e g 0 1) the results fully confirm those by Graham [2] As both Mach number and wave number are in creased however, the shape of the lift function becomes distorted and even forms a small loop near the frequency parameter yielding a so called transition region ' in which the gust convection velocity along the wave front exceeds the sonic speed

Turning to the cascade blades the present method has been checked against the two dimensional ($\lambda = 0$) cascade results by 1) Whitehead[12] for incompressible flow and 2) Lane and Friedman [13] and 3) Smith[14] for compressible flow Agree ment was good Consequently further calculations are carried out to look into the effect of the spanwise wave number λ choosing a typical cascade geometry (i e $s/c = 1$ and $\theta = 60$ deg) A peculiar phenomenon occurring in cascade flow is the acoustic resonance (see Appendix) Com pressibility introduces a pair of acoustic waves up and downstream of the cascade The resonance criterion postulates the condition among the parameters ω M_0 λ, s/c θ and σ for which those waves are propagating unattenuated or not It is given by

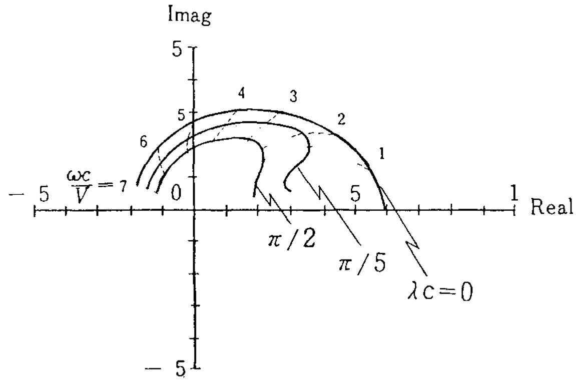
$$2\pi\nu - \sigma = \{-\mu tg\theta \pm \sqrt{(\beta^2 + tg^2\theta)(\kappa^2 - \lambda^2/\beta^2)}\}s\cos\theta$$

in which ν may be any integer and $0 \le \sigma \le 2\pi$ Figure 5 illustrates this relationship A point of notice here is that the introduction of λ cuts off the otherwise propagating acoustic

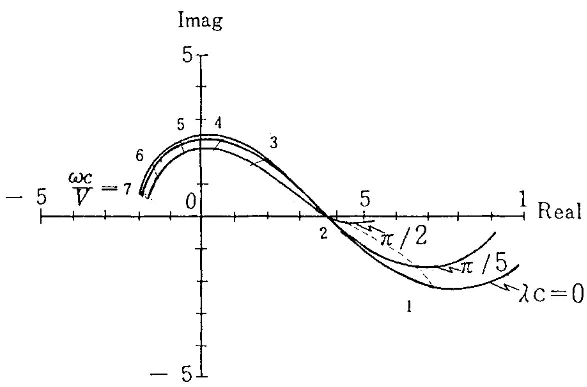


a) $M_0 = 0\ 7$ b) $\omega c/V = \pi/2$

Fig 5 Resonance criterion with various spanwise wave number λ

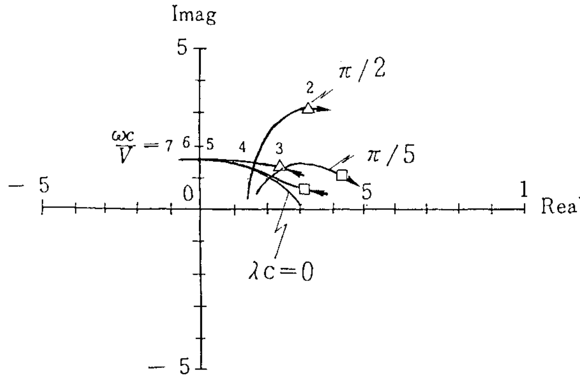


a) $\sigma = 0$ deg

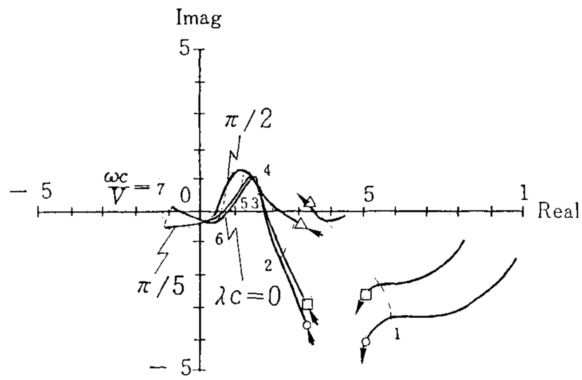


b) $\sigma = 180$ deg

Fig 6 Effect of spanwise wave number λ of wakes upon lift coef ficient for incompressible cascade flow



a) $\sigma = 0$ deg



b) $\sigma = 180$ deg

Fig 7 Effect of spanwise wave number λ of wakes upon lift coef ficient for subsonic cascade flow ($M_0 = 0\ 7$)

waves of low frequencies which are generated with the in terblade phase angle around $\sigma = 0$ Figures 6 and 7 illustrate the above mentioned circumstances in the problem of wake and cascade blade interaction The obliqueness of the wake with respect to the blade leading edge in relation to the parameter λ may be determined from the corresponding problem of a single airfoil Figure 6 compares the effect of λ for incompressible cascade flow with two kinds of σ The response curve corresponding to Sears' function thus differs greatly in shape, depending upon σ The modulus of the lift coefficient diminishes as λ is increased Figure 7 shows the results at Mach number $M_0 = 0\ 7$ Due to the presence of the acoustic resonance the response curves behave abruptly near the corresponding frequency The parameter λ is seen to be very influential upon the response curve in the low range of frequencies involving the acoustic resonance

Finally the case when airfoils are vibrating is studied Aerodynamic lift and moment are now the sum of those proportional to the vibration amplitude (h_λ α_λ) and those due to the gust/wake strength $\tilde{v}_\lambda$ (Note that the gust wave length $2\pi/\lambda$ may differ from the characteristic length $2\pi/\lambda$ of the vibration along the airfoil span) The vibration becomes significant when the excitation frequency ω coincides with one of the natural (or flutter) frequencies corresponding to bending and torsional airfoil motion The lift or moment due to gust/wake excitation should then be exactly balanced by the sum of mechanical and aerodynamic damping forces The ratios $|\omega h_\lambda / v_\lambda|$ and $|\alpha_\lambda V / v_\lambda|$ may be a measure of the gust effectiveness to excite the vibrations and thus will be called the excitation factor In the absence of mechanical damping, these factors are given by $|\mathcal{C}_{Lw}| / |\mathrm{Re}(\mathcal{C}_{Lq})|$ and $|\mathcal{C}_{Mw}| / |\mathrm{Im}(\mathcal{C}_{M\alpha})|$ where Re and Im mean the real and imaginary parts and the suffixes w q and α denote that the coefficients are nondimensionalized by the upwash velocities of gust/wake bending, and torsional vibrations respectively Figure 8 shows the plots of the logarithmic decrement γ_α for torsional vibration in cascade due to aerodynamic damping against the frequency parameter $\omega c/V$ The value γ_α should be positive in order to suppress flutter The axis of rotation x_η is assumed to be located at the midchord and the interblade phase angle σ is $\pi/4$ In the absence of mechanical damping therefore the frequency parameter at the onset of flutter is seen to be lowered as Mach number M_0 is increased con firming that compressibility effect is highly favorable This Mach number effect appears to be weaker when the spanwise modal dependence is taken into account as shown from the curves with λc being varied For incompressible flow however, the introduction of spanwise modes slightly decreases the flutter frequency It may thus be noticed that such a tendency is coupled with the presence of the resonance condition and therefore further studies will be needed in order to draw a precise conclusion The presence of the acoustic resonance greatly influences the excitation factors as well Calculations for bending vibration indicate that the plane mode gust/wake ($\lambda\ c = 0$) gives in general the largest ex

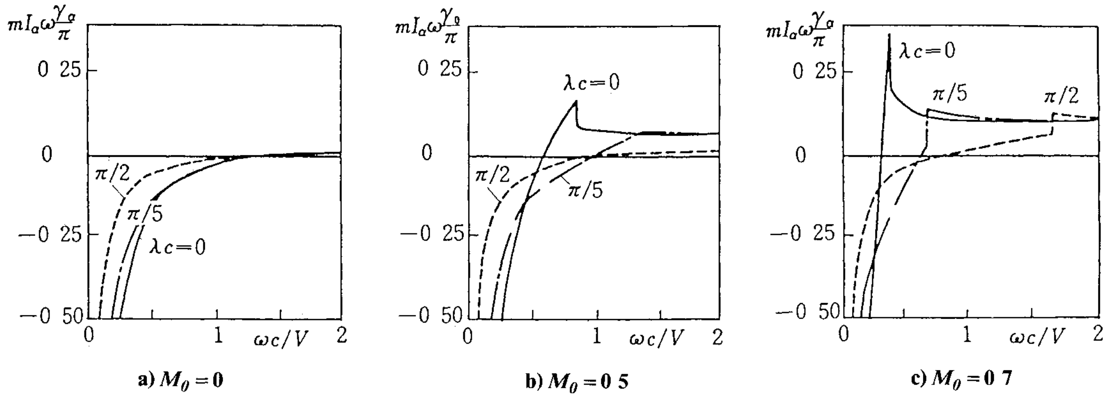


Fig 8 Effect of spanwise mode on aerodynamic damping for torsional vibration around midchord axis; $s/c=1$ $\theta=60$ deg $\sigma=45$ deg

citation factor for both single and cascade airfoils Close to the cascade resonance condition however this no longer holds and the largest excitation factor is often found there

Conclusions

A linearized theory has been developed to provide check results for advancing numerical calculations of the unsteady lift and moment of airfoils vibrating in subsonic flow with oblique sinusoidal gusts The airfoils are thin flat plates of infinite span and may be in cascade

Effects of Mach number and gust obliqueness in terms of spanwise wave number have been examined and the following principal results have been obtained:

1) The sinusoidal gust response curve of a single airfoil plotted on the real imaginary lift coefficient plane becomes greatly distorted in shape near the transition frequency at which the convection velocity along the gust wave front exceeds the sonic speed as the wave number and Mach number are increased

2) In the case of cascade blades the presence of the acoustic resonance causes an abrupt behavior of the response curve at the corresponding frequency and interblade phase angle Introduction of the spanwise gust wave parameter cuts off the otherwise propagating plane mode acoustic waves thereby having a greater effect on the difference in the gust response plots at the lower frequencies

3) The effect of increasing Mach number upon the instability of torsionally vibrating blades is found to be favorable This effect is however diminished when the spanwise modal dependence is taken into consideration

Appendix

Kernel Function for Pressure

The lift component of f normal to the airfoil surface is expressed by

$$f(x\ y\ z)=\left\langle\begin{matrix}-f_\lambda(x)\delta(y)\\ -\sum_m f_\lambda(x-ms\sin\theta)\delta(y-ms\cos\theta)e^{im\sigma}\end{matrix}\right\rangle \times\exp[i\omega t+i\lambda z] \tag{A1}$$

in which $\delta(y)$ is the Dirac delta function and the distribution f_λ extends from the leading to the trailing edges The upper row of the right side of Eq (A1) is for an isolated airfoil while the lower row gives the superposition of every mth airfoil effect in cascade (The reference airfoil is numbered as $m=0$) Substitution of the expression Eq (A1) into Eq (4) yields:

$$\mathcal{P}(x\ y|\lambda)=\frac{\partial}{\partial y}\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\frac{\exp[i\xi x-i\eta y]}{\beta^2(\xi-\mu)^2+\eta^2+\Omega^2}\mathrm{d}\xi\mathrm{d}\eta$$

$$\Omega^2\equiv\lambda^2-\beta^2\kappa^2 \tag{A2}$$

where $\mu=\kappa M_0$ and $\kappa=(M_0/\beta^2)(\omega/V)$ The outgoing wave condition fixes the proper branch of the integration According to whether the value of Ω^2 is positive or negative the double integral of Eq (A2) is reduced to different forms of analytic functions i e

$$\mathcal{P}(x\ y|\lambda)=\frac{\partial}{\partial y}\frac{e^{i\mu x}}{\beta}\left\langle\begin{matrix}\frac{1}{2\pi}K_0\left(\frac{\Omega}{\beta}r\right)\\ \frac{(-i)}{4}H_0^{(2)}\left(\frac{\sqrt{|\Omega^2|}}{\beta}r\right)\end{matrix}\right. \quad \text{as } \Omega^2\gtrless 0 \tag{A3}$$

where K_0 and $H_0^{(2)}$ are the modified Bessel function of the second kind of order 0 and the Hankel function of the second kind of order 0, respectively and $r^2=x^2+\beta^2y^2$ This is the expression of the kernel in Eq (6) for a single airfoil The marginal condition of $\Omega^2=0$ occurs when $\lambda=\beta\kappa$ or the inclination angle Λ of the gust propagating direction is given by $tg^{-1}[\lambda/(\omega/V)]=tg^{-1}(M_0/\beta)$ Since the gust is convected by the mean flow velocity V the convection velocity along the wave fronts becomes $V/\sin\Lambda=V/M_0=C_0$ i e just sonic speed When this happens the kernel is given by $\mathcal{P}(x\ y|\lambda)=-(\beta/2\pi)\exp(i\mu x)y/r^2$ In order to derive the expression of the kernel function for cascade airfoils it is convenient to apply the formula

$$\sum_m e^{imx}=2\pi\sum\delta(2\pi\nu-x) \tag{A4}$$

directly to the expression Eq (A2) The result is

$$\mathcal{P}(x\ y|\lambda)=\frac{\partial}{\partial y}\sum_m\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}$$

$$\times\frac{\exp[i\xi(x-ms\sin\theta)-i\eta(y-ms\cos\theta)+im\sigma]}{\beta^2(\xi-\mu)^2+\eta^2+\Omega^2}\mathrm{d}\xi\mathrm{d}\eta$$

$$=\frac{\partial}{\partial y}\sum\frac{(-)}{s\cos\theta}\frac{1}{2}\frac{1}{\sqrt{D}_\lambda}$$

$$\times\exp\left[-i\frac{2\pi\nu-\sigma}{s\cos\theta}y+A_\nu x+B_\lambda|x|\right] \tag{A5}$$

where $\chi = x - ytg\theta$

$$A = i\mu - i\frac{tg\theta}{\beta^2 + tg^2\theta}\left(\mu tg\theta + \frac{2\pi\nu - \sigma}{s\cos\theta}\right) \qquad B_\lambda = -\frac{\sqrt{D_{\nu\lambda}}}{\beta^2 + tg^2\theta}$$

and

$$D_\lambda = \beta^2\left(\mu tg\theta + \frac{2\pi\nu - \sigma}{s\cos\theta}\right)^2 + \Omega^2(\beta^2 + tg^2\theta)$$

Whether the solution is of decaying or propagating type depends upon the sign of the value D_λ Resonance occurs when $D_{\nu\lambda} = 0$ at which the kernel blows up and the lift therefore should diminish to zero Given the cascade geometry, there is a particular set of M_0 ω and σ for this resonance to happen

Kernel Function for Induced Velocity

Once the kernel function for pressure is known, the one for induced velocity normal to the airfoil surface is obtained by a straightforward integration:

$$\mathcal{V}(x\,y|\lambda) = -e^{-i\frac{\omega}{V}x}\int_{-\infty}^{x} e^{i\frac{\omega}{V}\tau}\frac{\partial}{\partial y}\mathcal{P}(\tau\,y|\lambda)\,d\tau \qquad (A6)$$

For a single airfoil on the airfoil surface this reduces to:

$$\mathcal{V}(x\,0|\lambda) = \mathcal{V}_0(x\,0|\lambda) + \mathcal{V}_I(x\,0|\lambda) \qquad (A7)$$

where

$$\mathcal{V}_0(x\,0|\lambda) = -\beta e^{i\mu x}\left[\frac{\partial}{\partial x} - i\frac{1}{\beta^2}\frac{\omega}{V}\right] \times \left\langle \begin{matrix} \frac{(-)}{2\pi}K_0\left(\frac{\Omega}{\beta}|x|\right) \\ \frac{i}{4}H_0^{(2)}\left(\frac{\sqrt{|\Omega^2|}}{\beta}|x|\right)\end{matrix}\right. \qquad (A8)$$

$$\mathcal{V}_I(x\,0|\lambda) = \left\{\frac{1}{\beta^2}\left(\frac{\omega}{V}\right)^2 + \Omega^2\right\}\frac{1}{\beta}e^{-i\frac{\omega}{V}x}$$

$$\times\left[\int_0^x e^{i\frac{1}{\beta^2}\frac{\omega}{V}\tau}\left\langle \begin{matrix} \frac{(-)}{2\pi}K_0\left(\frac{\Omega}{\beta}|\tau|\right) \\ \frac{i}{4}H_0^{(2)}\left(\frac{\sqrt{|\Omega^2|}}{\beta}|\tau|\right)\end{matrix}\right\rangle d\tau\right.$$

$$\left. - \frac{\beta}{2\pi}\frac{1}{\sqrt{|\Omega^2|}}\left\langle \begin{matrix} \frac{1}{\sqrt{a^2+1}}\left\{\frac{\pi}{2} - i\ln|a + \sqrt{a^2+1}|\right\} \\ \frac{(-i)}{\sqrt{a^2+1}}\ln|a + \sqrt{a^2-1}| \end{matrix}\right\rangle\right] \qquad (A9)$$

The upper row applies for $\Omega^2 > 0$ while the lower for $\Omega^2 < 0$ and

$$a = \frac{1}{\beta^2}\frac{\omega}{V}\Big/\frac{\sqrt{|\Omega^2|}}{\beta}$$

The limit of $\lambda = 0$ naturally yields the expression for a vibrating two dimensional airfoil

$$\mathcal{V}(x\,0|0) = -\frac{i\beta}{4}e^{i\mu x}\left[\frac{\partial}{\partial x} - i\frac{1}{\beta^2}\frac{\omega}{V}\right]$$

$$\times H_0^{(2)}(\kappa|x|) + \left(\frac{\omega}{V}\right)^2\frac{e^{-\frac{\omega}{V}x}}{\beta}\left[\frac{i}{4}\int_0^x e^{i\frac{1}{\beta^2}\frac{\omega}{V}\tau}\right.$$

$$\left.\times H_0^{(2)}(\kappa|\tau|)\,d\tau + \frac{i}{2\pi\kappa}\frac{M}{\beta}\ln\frac{1+\beta}{M}\right] \qquad (A10)$$

For the particular case of $\Omega^2 = 0$ it is found that

$$\mathcal{V}(x\,0|\lambda) = -\frac{1}{2\pi}\frac{e^{i\mu x}}{x} + i\frac{1}{\beta^2}\frac{\omega}{V}\frac{e^{-i(\omega/V)x}}{2\pi}$$

$$\times\left\{C_i\left(\frac{1}{\beta^2}\frac{\omega}{V}x\right) + iS_i\left(\frac{1}{\beta^2}\frac{\omega}{V}x\right) + i\frac{\pi}{2}\right\} \qquad (A11)$$

where C_i and S_i are the cosine and sine integrals

The corresponding expression for cascade airfoils is obtained by substituting Eq (A5) into Eq (A6) The result is

$$\mathcal{V}(x\,0|\lambda) = \frac{1}{\beta^2 + tg^2\theta}\sum_\nu \frac{1}{s\cos\theta}[K_{\nu\lambda}^- e^{-i\xi^- x}H(-x)$$

$$-\{K_{\nu\lambda}^+ e^{-i\xi^+ x} + K_{\nu\lambda}e^{-i\frac{\omega}{V}x}\}H(x)] \qquad (A12)$$

where

$$K_{\nu\lambda}^- = \frac{\beta^2(\xi^- + \mu)^2 + \Omega^2}{(\xi^- - \xi^+)(\xi^- - \omega/V)}$$

$$K_{\nu\lambda}^+ = \frac{\beta^2(\xi^+ + \mu)^2 + \Omega^2}{(\xi^+ - \xi^-)(\xi^+ - \omega/V)}$$

$$K_{\nu\lambda} = \frac{(1/\beta^2)(\omega/V)^2 + \Omega^2}{(\xi^+ - \omega/V)(\xi^- - \omega/V)}$$

$H(x)$ is the step function and $\xi^+ = iA + iB_{\nu\lambda}$ and $\xi^- = iA_\nu - iB_\lambda$ the same symbols as are used in Eq (A5) The series summation otherwise exponentially convergent, is hampered by the last term of Eq (A12) which is the contribution of the shed vortex waves In order to carry out the numerical calculation the asymptotic form as $|\nu|$ becomes large is subtracted from the latter and summed up analytically

$$\mathcal{V}(x\,0|\lambda) = \frac{(-)}{\beta^2 + tg^2\theta}\frac{e^{-i(\omega/V)x}}{s\cos\theta}H(x)\left[K_{0\lambda}^\circ\right.$$

$$+\sum_{\nu\neq 0}\{K_{\nu\lambda} - K_\lambda|_{\nu\,\text{large}}\} + \left\{\frac{1}{\beta^2}\left(\frac{\omega}{V}\right)^2 + \Omega^2\right\}$$

$$\left.\times(\beta^2 + tg^2\theta)\frac{s\cos\theta}{2\sqrt{\Omega^2}}\left\{\coth\left(\sqrt{\Omega^2}\frac{s\cos\theta}{2}\right) - \frac{2}{\sqrt{\Omega^2}s\cos\theta}\right\}\right] \qquad (A13)$$

In this way the induced velocity on the airfoil surface can be calculated

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